Matrices and Vector Spaces With Complex Scalars

1. Matrices With Complex Numbers

E.g. 1 Solve the linear system

$$z_1 - z_2 + (1+i)z_3 = i$$
$$iz_1 - 2iz_2 + iz_3 = 2-i$$
$$iz_2 - (1+i)z_3 = 1+2i$$

Solution:

$$\left[\begin{array}{ccc|c} 1 & -1 & 1+i & i \\ i & -2i & i & 2-i \\ 0 & i & -(1+i) & 1+2i \end{array}\right]$$

$R_2 - iR_1$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 1+i & i \\ 0 & -i & 1 & 3-i \\ 0 & i & -1-i & 1+2i \end{array}\right]$$

$i - i = 0$
$-2i - (-i) = -2i + i = -i$
$i - i(1+i) = i - (i + i^2) = i - i - (-1) = 1$
$2 - i - i^2 = 2 - i - (-1) = 3 - i$

$R_3 + R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 1+i & i \\ 0 & -i & 1 & 3-i \\ 0 & 0 & -i & 4+i \end{array}\right]$$

$R_2 \cdot i,\ R_3 \cdot i$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 1+i & i \\ 0 & 1 & i & 1+3i \\ 0 & 0 & 1 & -1+4i \end{array}\right]$$

$R_1 + R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 1+2i & 1+4i \\ 0 & 1 & i & 1+3i \\ 0 & 0 & 1 & -1+4i \end{array}\right]$$

$r_2 - ir_3$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 1+2i & 1+4i \\ 0 & 1 & 0 & 5+4i \\ 0 & 0 & 1 & -1+4i \end{array}\right]$$

$r_1 - r_3$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 2i & 2 \\ 0 & 1 & 0 & 5+4i \\ 0 & 0 & 1 & -1+4i \end{array}\right]$$

$r_1 - 2iR_3$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 10+2i \\ 0 & 1 & 0 & 5+4i \\ 0 & 0 & 1 & -1+4i \end{array}\right]$$

E.g. 2 Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 2i & 1+i \\ 1 & 3i & i \\ 0 & 1+i & -1 \end{bmatrix}$$

Solution

$$\left[\begin{array}{ccc|ccc} 1 & 2i & 1+i & 1 & 0 & 0 \\ 1 & 3i & i & 0 & 1 & 0 \\ 0 & 1+i & -1 & 0 & 0 & 1 \end{array}\right]$$

$R2 - R1$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2i & 1+i & 1 & 0 & 0 \\ 0 & i & -1 & -1 & 1 & 0 \\ 0 & 1+i & -1 & 0 & 0 & 1 \end{array}\right]$$

$R2 \cdot (-i)$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2i & 1+i & 1 & 0 & 0 \\ 0 & 1 & i & i & -i & 0 \\ 0 & 1+i & -1 & 0 & 0 & 1 \end{array}\right]$$

$R_1 - 2iR_2$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3+i & 3 & -2 & 0 \\ 0 & 1 & i & i & -i & 0 \\ 0 & 1+i & -1 & 0 & 0 & 1 \end{array}\right]$$

$R_3 - (1+i)R_2$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3+i & 3 & -2 & 0 \\ 0 & 1 & i & i & -i & 0 \\ 0 & 0 & -i & 1-i & -1+i & 1 \end{array}\right]$$

$R_2 + R_3$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3+i & 3 & -2 & 0 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & -i & 1-i & -1+i & 1 \end{array}\right]$$

$R_1 + R_3$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 4-i & -3+i & 1 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & -i & 1-i & -1+i & 1 \end{array}\right]$$

$R_3 \cdot (i)$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 4-i & -3+i & 1 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1+i & -1-i & i \end{array}\right]$$

$R_1 - 3R_3$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1-4i & 4i & 1-3i \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1+i & -1-i & i \end{array}\right]$$

$$\therefore \quad A^{-1} = \begin{bmatrix} 1-4i & 4i & 1-3i \\ 1 & -1 & 1 \\ 1+i & -1-i & i \end{bmatrix}$$

Theorems and Definitions:

1. Conjugate Transpose / Hermitian Adjoint:

Let $A = [a_{ij}]$ be a $m \times n$ matrix with complex scalar entries. Then,

- The conjugate of A is the $m \times n$ matrix $\bar{A} = [\bar{a}_{ij}]$.
- The conjugate transpose or Hermitian adjoint of A is the matrix $A^* = [\bar{a}_{ij}]^T$

E.g. 3 Find the conjugate transpose of the matrix

$$A = \begin{bmatrix} 1 & i & 1+i \\ 2 & 0 & i \\ 2i & 1 & 1-i \end{bmatrix}$$

Solution

1. Find the transpose of A

$$\begin{bmatrix} 1 & 2 & 2i \\ i & 0 & 1 \\ 1+i & i & 1-i \end{bmatrix}$$

2. Find the conjugate of each element

$$\begin{bmatrix} 1 & 2 & -2i \\ -i & 0 & 1 \\ 1-i & -i & 1+i \end{bmatrix}$$

Recall, the conjugate of $a+ib$ is $a-ib$.

$$\therefore A^* = \begin{bmatrix} 1 & 2 & -2i \\ -i & 0 & 1 \\ 1-i & -i & 1+i \end{bmatrix}$$

Properties of the Conjugate Transpose:

Let A and B be $m \times n$ matrices. Then,

1. $(A^*)^* = A$
2. $(A+B)^* = A^* + B^*$
3. $(zA)^* = \bar{z}A^*$ for any scalar $z \in \mathbb{C}$.
4. If A and B are square matrices, $(AB)^* = B^* A^*$

Note: If A is a $n \times n$ matrix, then $(A+A^*)^* = A+A^*$

Proof: $(A+A^*)^*$

$= A^* + (A^*)^*$

$= A^* + A$

$= A + A^*$

2. Unitary Matrix:

A square matrix with complex entries is a unitary matrix if its column vectors are orthogonal unit vectors. I.e. $U^*U = I$ A real unitary matrix is an orthogonal matrix.

3. Hermitian Matrix:

A square matrix is Hermitian if $H^* = H$. A real Hermitian matrix is a symmetric matrix.

4. Basis of Complex Number Vector Spaces:

C^n has the same standard basis as R^n. That basis is $\{e_1, e_2, \ldots e_n\}$. However, the field of scalars is C.

E.g. 4 Determine whether the set $S = \{[1, 2i, 1+i], [1, 3i, i], [0, 1+i, -1]\}$ is independent and a basis for C^3.

Solution:

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 2i & 3i & 1+i & 0 \\ 1+i & i & -1 & 0 \end{array}\right]$$

$R_2 \cdot -i$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 2 & 3 & 1-i & 0 \\ 1+i & i & -1 & 0 \end{array}\right]$$

$R_2 - 2R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1-i & 0 \\ 1+i & i & -1 & 0 \end{array}\right]$$

$R_3 - (1+i)R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1-i & 0 \\ 0 & -1 & -1 & 0 \end{array}\right]$$

$R_3 \times (-i)$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1-i & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

$R_2 - (1-i)R_3$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

$R_2 \times (-i)$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$\therefore$ The vectors are linearly independent and hence, a basis for C^3.

5. Coordinate Vectors in Complex Number Vector Spaces:

E.g. 5 Find the coordinate vector v_B in $\mathbb{C}^3$ of the vector $v = [i, 2-i, 1+2i]$ relative to the ordered basis
$B = ([1, i, 0], [-1, -2i, i], [1+i, i, -1-i])$.

Solution:

$$\left[\begin{array}{ccc|c} 1 & -1 & 1+i & i \\ i & -2i & i & 2-i \\ 0 & i & -1-i & 1+2i \end{array}\right]$$

$R_1 + R_3$

$$\sim \left[\begin{array}{ccc|c} 1 & -1+i & 0 & 1+3i \\ i & -2i & i & 2-i \\ 0 & i & -(1+i) & 1+2i \end{array}\right]$$

$R_2 \cdot (-i)$

$$\sim \left[\begin{array}{ccc|c} 1 & -1+i & 0 & 1+3i \\ 1 & -2 & 1 & -1-2i \\ 0 & i & -(1+i) & 1+2i \end{array}\right]$$

$R_2 - R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & -1+i & 0 & 1+3i \\ 0 & -1-i & 1 & -2-5i \\ 0 & i & -(1+i) & 1+2i \end{array}\right]$$

$R_3 + (1+i)R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & -1+i & 0 & 1+3i \\ 0 & -1-i & 1 & -2-5i \\ 0 & -i & 0 & 4-5i \end{array}\right]$$

$R_3 \times (i)$

$$\sim \left[\begin{array}{ccc|c} 1 & -1+i & 0 & 1+3i \\ 0 & -1-i & 1 & -2-5i \\ 0 & 1 & 0 & 5+4i \end{array}\right]$$

$R_2 \longleftrightarrow R_3$

$$\sim \left[\begin{array}{ccc|c} 1 & -1+i & 0 & 1+3i \\ 0 & 1 & 0 & 5+4i \\ 0 & -1+i & 1 & -2-5i \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 10+2i \\ 0 & 1 & 0 & 5+4i \\ 0 & 0 & 1 & -1+4i \end{array}\right]$$

$$V_B = \begin{bmatrix} 10+2i \\ 5+4i \\ -1+4i \end{bmatrix}$$

Inner Product with Complex Numbers:

Let $u = [u_1, u_2, \ldots u_n]$ be vectors in C^n.
Let $v = [v_1, v_2, \ldots v_n]$ be vectors in C^n.

Then, $\langle u, v\rangle = \bar{u}_1 v_1 + \bar{u}_2 v_2 + \ldots + \bar{u}_n v_n$

Properties:

1. $\langle u,u\rangle = |u_1|^2 + |u_2|^2 + \ldots + |u_n|^2$
2. $\langle u,u\rangle \geq 0$, and $\langle u,u\rangle = 0$ iff $u=0$
3. $\langle u,v\rangle = \overline{\langle v,u\rangle}$ I.e. The inner product in C^n is not commutative.
4. $\langle w, (u+v)\rangle = \langle w,u\rangle + \langle w,v\rangle$
5. $\langle zu, v\rangle = \bar{z}\langle u,v\rangle$
6. $\langle u, zv\rangle = z\langle u,v\rangle$

Proofs:

1. $\langle u,u\rangle = \bar{u}_1 u_1 + \bar{u}_2 u_2 + \ldots$
$= |u_1|^2 + |u_2|^2 + \ldots$ (Recall, $(a+ib)(a-ib) = a^2+b^2$).

2. From 1, we see that $\langle u,u\rangle$ is always non-negative and only equals to 0 iff $u_1 = u_2 = \ldots = 0$.

3. $\langle u,v\rangle = \bar{u}_1 v_1 + \bar{u}_2 v_2 + \ldots$
$= v_1 \bar{u}_1 + v_2 \bar{u}_2 + \ldots$
$= \overline{\bar{v}_1 u_1} + \overline{\bar{v}_2 u_2} + \ldots$
$= \overline{\langle v,u\rangle}$

4. $\langle w, (u+v)\rangle = \bar{w}_1 (u_1+v_1) + \bar{w}_2(u_2+v_2) + \ldots$
$= \bar{w}_1 u_1 + \bar{w}_1 v_1 + \bar{w}_2 u_2 + \bar{w}_2 v_2 + \ldots$
$= (\bar{w}_1 u_1 + \bar{w}_2 u_2 + \ldots) + (\bar{w}_1 v_1 + \bar{w}_2 v_2 + \ldots)$
$= \langle w,u\rangle + \langle w,v\rangle$

5. $\langle zu, v\rangle = (\overline{zu_1})v_1 + (\overline{zu_2})v_2 + \ldots$
$= \bar{z}\,\bar{u}_1 v_1 + \bar{z}\,\bar{u}_2 v_2 + \ldots$
$= \bar{z}\,(\bar{u}_1 v_1 + \bar{u}_2 v_2 + \ldots)$
$= \bar{z}\langle u, v\rangle$

6. $\langle u, zv\rangle = (\bar{u}_1)(zv_1) + (\bar{u}_2)(zv_2) + \ldots$
$= z\bar{u}_1 v_1 + z\bar{u}_2 v_2 + \ldots$
$= z\,(\bar{u}_1 v_1 + \bar{u}_2 v_2 + \ldots)$
$= z\langle u, v\rangle$

Note:

1. The magnitude or norm of a vector v in C^n is
$\|v\| = \sqrt{\langle v,v\rangle} = \sqrt{|v_1|^2 + |v_2|^2 + \ldots}$

2. 2 vectors in C^n are orthogonal if their inner product equals to 0. I.e. u and v are orthogonal if $\langle u,v\rangle = 0$.

3. Vectors of magnitude 1 are unit vectors.

4. 2 vectors are parallel if one equals to a scalar multiple of the other. I.e. u is parallel to v if $u = zv$.

E.g. 6. Find a unit vector parallel to $[1, i, 1+i]$.

Solution:
$\sqrt{(1)(1) + (-i)(i) + (1-i)(1+i)}$

$= \sqrt{1+1+2}$

$= 2$

$\therefore \pm\frac{1}{2}[1, i, 1+i]$ are the solutions

Gram-Schmidt with Complex Numbers:

Let u and v be vectors in $\mathbb{C}^n$.
Then, the formula for Gram-Schmidt is

$$v_1 = a_1$$

$$v_2 = a_2 - \frac{\langle v_1, a_2 \rangle}{\langle v_1, v_1 \rangle} v_1$$

$$v_n = a_n - \left(\frac{\langle v_1, a_n \rangle}{\langle v_1, v_1 \rangle} v_1 + \dots + \frac{\langle v_{n-1}, a_n \rangle}{\langle v_{n-1}, v_{n-1} \rangle} v_{n-1} \right)$$